How to determine where the polar curves $r = f(\theta)$ and $r = g(\theta)$ intersect using $r = \cos 2\theta$ and $r = 1 + \cos 2\theta$ as an example

NOTE: This technique only gives a complete list of intersection points if the periods of $f(\theta)$ and $g(\theta)$ both have the form $\frac{2\pi}{n}$ where n is an integer (may be different values of n for each function)

NOTE: Both functions in our example have a period of $\frac{2\pi}{2}$

1. Solve $f(\theta) = 0$ and $g(\theta) = 0$ separately for $\theta \in [0, 2\pi]$.
   If both equations have solutions, then the graphs intersect at the pole (though not necessarily at the same value of θ).

$$\cos 2\theta = 0$$
$$0 \le \theta \le 2\pi \quad \Rightarrow \quad 0 \le 2\theta \le 4\pi$$
$$2\theta = \tfrac{\pi}{2}, \tfrac{3\pi}{2}, \tfrac{5\pi}{2}, \tfrac{7\pi}{2} \quad \Rightarrow \quad \theta = \tfrac{\pi}{4}, \tfrac{3\pi}{4}, \tfrac{5\pi}{4}, \tfrac{7\pi}{4}$$

$$1 + \cos 2\theta = 0 \quad \Rightarrow \quad \cos 2\theta = -1$$
$$0 \le \theta \le 2\pi \quad \Rightarrow \quad 0 \le 2\theta \le 4\pi$$
$$2\theta = \pi, 3\pi \quad \Rightarrow \quad \theta = \tfrac{\pi}{2}, \tfrac{3\pi}{2}$$

Both curves pass through the pole, so they intersect at the pole

2. Solve $f(\theta) = g(\theta)$ for $\theta \in [0, 2\pi]$.
   If $\theta = \theta_1$, then the graphs intersect at the point $(r, \theta) = (f(\theta_1), \theta_1) = (g(\theta_1), \theta_1)$.

$$\cos 2\theta = 1 + \cos 2\theta \quad \Rightarrow \quad 0 = 1$$

No intersection where polar co-ordinates are the same on both curves

3. Rewrite $r = f(\theta)$ by substituting $(r, \theta) = (-r, \pi + \theta)$.
   That is, $r = f(\theta)$ becomes $-r = f(\pi + \theta)$ ie. $r = -f(\pi + \theta)$.
   Solve $-f(\pi + \theta) = g(\theta)$ for $\theta \in [0, \pi]$.
   If $\theta = \theta_2$, then the graphs intersect at
   the point $(r, \theta) = (f(\pi + \theta_2), \pi + \theta_2)$ on the graph of $r = f(\theta)$
   and the point $(r, \theta) = (g(\theta_2), \theta_2)$ on the graph of $r = g(\theta)$.
   (These are the same point with different polar coordinates.)

$$-r = \cos 2(\pi + \theta) \quad \Rightarrow \quad r = -\cos(2\pi + 2\theta) = -\cos 2\theta$$
$$-\cos 2\theta = 1 + \cos 2\theta \quad \Rightarrow \quad \cos 2\theta = -\tfrac{1}{2}$$
$$0 \le \theta \le \pi \quad \Rightarrow \quad 0 \le 2\theta \le 2\pi$$
$$2\theta = \tfrac{2\pi}{3}, \tfrac{4\pi}{3} \quad \Rightarrow \quad \theta = \tfrac{\pi}{3}, \tfrac{2\pi}{3}$$

The curves intersect at 2 points

At $\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$ on $r = \cos 2\theta$, $r = \cos 2(\frac{4\pi}{3}) = \cos\frac{8\pi}{3} = -\frac{1}{2}$ and
at $\theta = \frac{\pi}{3}$ on $r = 1 + \cos 2\theta$, $r = 1 + \cos 2(\frac{\pi}{3}) = 1 + \cos\frac{2\pi}{3} = \frac{1}{2}$
$(-\frac{1}{2}, \frac{4\pi}{3})$ and $(\frac{1}{2}, \frac{\pi}{3})$ are different polar coordinates for the same point

At $\theta = \pi + \frac{2\pi}{3} = \frac{5\pi}{3}$ on $r = \cos 2\theta$, $r = \cos 2(\frac{5\pi}{3}) = \cos\frac{10\pi}{3} = -\frac{1}{2}$ and
at $\theta = \frac{2\pi}{3}$ on $r = 1 + \cos 2\theta$, $r = 1 + \cos 2(\frac{2\pi}{3}) = 1 + \cos\frac{4\pi}{3} = \frac{1}{2}$
$(-\frac{1}{2}, \frac{5\pi}{3})$ and $(\frac{1}{2}, \frac{2\pi}{3})$ are different polar coordinates for the same point

4. Rewrite $r = g(\theta)$ by substituting $(r, \theta) = (-r, \pi + \theta)$.
That is, $r = g(\theta)$ becomes $-r = g(\pi + \theta)$ ie. $r = -g(\pi + \theta)$.
Solve $f(\theta) = -g(\pi + \theta)$ for $\theta \in [0, \pi]$.
If $\theta = \theta_3$, then the graphs intersect at
the point $(r, \theta) = (f(\theta_3), \theta_3)$ on the graph of $r = f(\theta)$
and the point $(r, \theta) = (g(\pi + \theta_3), \pi + \theta_3)$ on the graph of $r = g(\theta)$.
(These are the same point with different polar coordinates.)

$$-r = 1 + \cos 2(\pi + \theta) \quad \Rightarrow \quad r = -1 - \cos(2\pi + 2\theta) = -1 - \cos 2\theta$$
$$\cos 2\theta = -1 - \cos 2\theta \quad \Rightarrow \quad \cos 2\theta = -\tfrac{1}{2}$$
$$0 \le \theta \le \pi \quad \Rightarrow \quad 0 \le 2\theta \le 2\pi$$
$$2\theta = \tfrac{2\pi}{3}, \tfrac{4\pi}{3} \quad \Rightarrow \quad \theta = \tfrac{\pi}{3}, \tfrac{2\pi}{3}$$

The curves intersect at 2 points

At $\theta = \frac{\pi}{3}$ on $r = \cos 2\theta$, $r = \cos 2(\frac{\pi}{3}) = \cos \frac{2\pi}{3} = -\frac{1}{2}$ and
at $\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$ on $r = 1 + \cos 2\theta$, $r = 1 + \cos 2(\frac{4\pi}{3}) = 1 + \cos \frac{8\pi}{3} = \frac{1}{2}$
$(-\frac{1}{2}, \frac{\pi}{3})$ and $(\frac{1}{2}, \frac{4\pi}{3})$ are different polar coordinates for the same point

At $\theta = \frac{2\pi}{3}$ on $r = \cos 2\theta$, $r = \cos 2(\frac{2\pi}{3}) = \cos \frac{4\pi}{3} = -\frac{1}{2}$ and
at $\theta = \pi + \frac{2\pi}{3} = \frac{5\pi}{3}$ on $r = 1 + \cos 2\theta$, $r = 1 + \cos 2(\frac{5\pi}{3}) = 1 + \cos \frac{10\pi}{3} = \frac{1}{2}$
$(-\frac{1}{2}, \frac{2\pi}{3})$ and $(\frac{1}{2}, \frac{5\pi}{3})$ are different polar coordinates for the same point

The lighter graph below is $r = \cos 2\theta$.
The darker graph below is $r = 1 + \cos 2\theta$.
The dots are the intersection points, and the numbers next to them are the step number (in the process above) at which those points were found.